Combined convection in non-Newtonian fluids along a nonisothermal vertical plate in a porous medium

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Abstract *The problem of combined convection from vertical surfaces in a porous medium saturated with power-law type non-Newtonian fluids along a non-isothermal vertical plate is investigated. The transformed conservation laws were solved numerically for the case of variable surface heat flux conditions. Results for the details of the velocity and temperature fields as well as the Nusselt number have been presented.*

Nomenclature

d = particle diameter
f = dimensionless stream function
g = acceleration due to gravity
k = permeability coefficient of the porous medium
K = thermal conductivity
n = viscosity index
Nu = Nusselt number
Pe = Peclet number
Ra = Rayleigh number
T = temperature
u, v = velocity components in x and y directions
x, y = axial and normal coordinates
α = thermal diffusivity
β = coefficient of thermal expansion
η = dimensionless distance
θ = dimensionless temperature
ν = kinematics viscosity
μ = dynamic viscosity
ρ = density
ε = porosity
τ_w = wall shear stress
χ = mixed convection nonsimilar parameter
ψ = stream function

Subscripts
w = surface conditions
∞ = free stream conditions

Introduction

The study of combined convection boundary layer flow along a vertical surface embedded in fluid-saturated porous media has received considerable interest recently. The interest in such studies was motivated by numerous thermal engineering applications in several areas such as geothermal engineering, thermal insulation systems, petroleum recovery, filtration processes, packed bed reactors, sensible heat storage beds, ceramic processing and ground water pollution.

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Similarity solutions for free convective heat transfer from a vertical plate in a fluid-saturated porous media were obtained by Cheng and Minkowycz (1977), Gorla and co-workers (Gorla and Zinalabedini, 1987; Gorla and Tornabene, 1988) developed a procedure to investigate the nonsimilar boundary layer problem of free convection from a vertical plate embedded in a porous medium with an arbitrarily varying surface temperature or heat flux. The problem of mixed convection from surfaces embedded in porous media was studied by Minkowycz *et al.* (1985) as well as Ranganathan and Viskanta (1984). Hsieh *et al.* (1993) presented nonsimilar solutions for mixed convection in porous media. Nakayama and Pop (1985) presented similarity solutions for the free, forced and combined convection. Kumari and Gorla (1997) examined the combined convection along a non-isothermal vertical plate in a porous medium. All these studies were concerned with Newtonian fluid flows. A number of industrially important fluids, including fossil fuels which may saturate underground beds, display non-Newtonian behavior. Non-Newtonian fluids exhibit a nonlinear relationship between shear stress and shear rate.

Chen and Chen (1988) presented similarity solutions for free convection of non-Newtonian fluids over vertical surfaces in porous media. Nakayama and Koyama (1991) studied the natural convection over a non-isothermal body of arbitrary shape embedded in a porous medium.

The present work has been undertaken in order to analyze the problem of nonsimilar mixed convection from a vertical non-isothermal flat plate embedded in non-Newtonian fluid saturated porous media. The boundary condition of variable surface temperature is treated in this paper. We have taken into account the presence of internal heat generation/absorption. The power law model of Ostwald-de-Waele, which is adequate for many non-Newtonian fluids, will be considered here. The transformed boundary layer equations are solved using a finite difference method. The numerical results for the velocity and temperature fields are obtained.

Analysis

Let us consider mixed convection from a heated vertical plate in a non-Newtonian fluid-saturated porous medium. The surface of the plate is maintained at a heat flux $q_w(x)$. The flow velocity and the pores of the porous medium are assumed to be small and therefore Darcy's model is assumed to be valid. The governing equations under Boussinesq and boundary layer approximation may be written as:

$$\frac{\partial u}{\partial x} + \frac{\partial \nu}{\partial y} = 0, \tag{1}$$

$$n u^{n-1} \frac{\partial u}{\partial y} = \left(\frac{\rho k g \beta}{\mu} \right) \frac{\partial T}{\partial y} \tag{2}$$

$$u \frac{\partial T}{\partial x} + \nu \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + Q(T - T_\infty) \tag{3}$$

In the above equations, u and v are the Darcian velocity components in x and y directions; T the temperature; n the viscosity index; ρ the density; μ the viscosity; β the volumetric coefficient of expansion; k the permeability of the porous medium and α the equivalent thermal diffusivity of the porous medium.

Non-Newtonian fluids generally exhibit a nonlinear relation between shear stress and shear rate. These fluids may be classified as inelastic and viscoelastic. The inelastic fluids may be subdivided as time-dependent fluids and time-independent fluids. The time-dependent fluids, in turn, are subdivided into two groups: thixotropic and rheopectic. The time-independent fluids can be subdivided into four groups: pseudoplastic, dilatant, Bingham plastic and pseudoplastic with yield stress.

Inelastic time-independent non-Newtonian fluids have received the greatest attention from rheologists, which has resulted in the development of a number of equations or models proposed to represent their flow behavior. The Ostwald-de-Waele power-law model represents several inelastic time-independent non-Newtonian fluids of practical interest and therefore has been used in this paper. When $n < 1$, the model describes pseudoplastic behavior, whereas $n > 1$ represents dilatant behavior.

Christopher and Middleman (1965) were the first to propose the form of Darcy law applicable to power-law fluids. In essence, the modified Darcy law as obtained by them can be written in vector notation:

$$\nabla p = \rho g - \left(\frac{\mu|\nu|^{n-1}}{k}\right)\mathbf{v}$$

where μ is the consistency of the power-law fluid and k is the modified permeability.

For the power law model of Ostwald-de-Waele, Christopher and Middleman (1965) and Dharmadhikari and Kale (1985) proposed the following relationships for the permeability:

$$k = \begin{cases} \dfrac{6}{25}\left(\dfrac{n\varepsilon}{3n+1}\right)^{n}\left[\dfrac{\varepsilon d}{3(1-\varepsilon)}\right]^{n+1} \\ \dfrac{2}{\varepsilon}\left[\dfrac{d\varepsilon^{2}}{8(1-\varepsilon}\right]^{n+1}\left(\dfrac{6n+1}{10n-3}\right)\left(\dfrac{16}{75}\right)^{3(10n-3)/(10n+11)} \end{cases} \tag{4}$$

In the above equation, d is the particle diameter and ε the porosity.

The appropriate boundary conditions are given by

$$\begin{aligned} & y = 0, \nu = 0, q_w = ax^{\lambda} \\ & y \to \infty : u = U_{\infty}, T = T_{\infty} \end{aligned} \tag{5}$$

where α and λ are constants. We note that $\lambda = 0$ corresponds to uniform heat flux wall conditions.

The continuity equation is automatically satisfied by defining a stream function $\psi(x, y)$ such that

$$u = \frac{\partial \psi}{\partial y} \text{and } \nu = -\frac{\partial \psi}{\partial x} \tag{6}$$

Proceeding with the analysis, we define the following transformations:

$$\begin{aligned}
\eta &= \left(\frac{y}{x}\right)\frac{\sqrt{Pe_x}}{\chi}, \\
\chi^{-1} &= 1 + \left[Ra_x^{*n} / Pe_x^{\overline{(2n+1)2}}\right]^{\frac{1}{(2n+1)}} \\
\psi &= \alpha \frac{\sqrt{Pe_x}}{\chi} f(\chi, \eta), \\
\theta &= \frac{(T - T_\infty) Pe_x^{1/2} \chi^{-1}}{\left(\frac{q_w x}{K}\right)} \\
Pe_x &= \frac{U_\infty x}{\alpha}, \\
Ra_x^* &= \frac{x}{\alpha}\left[\frac{\rho k g \beta (q_w x)}{\mu K}\right]^{\frac{1}{n}}, \\
H &= \frac{Q x^2}{\alpha Pe_x}
\end{aligned} \tag{7}$$

Substituting the expression in (7) into (2) and (3), the transformed governing equations may be written as:

$$n(f')^{n-1} f'' = (1 - \chi)^{2n+1} \theta'$$

$$\theta'' + H\chi^2\theta + \frac{1}{2}\left[+\left(\frac{\lambda+1}{2n+1}\right)(1-\chi)\right] f\theta' - \left(\frac{2\lambda+1}{2}\right)\left[1 - \frac{1}{2n+1}(1-\chi)\right] f'\theta \tag{8}$$

$$= \frac{2\lambda+1}{2(2n+1)}\chi(1-\chi)\left[\theta' \frac{\partial f}{\partial \chi} - f' \frac{\partial \theta}{\partial \chi}\right]$$

$$\left[\frac{1}{2} + \frac{2\lambda+1}{2n+1}(1-\chi)\right] f(\chi, 0) + \frac{2\lambda+1}{2n+1}\chi(\chi-1)\frac{\partial f}{\partial \chi}(\chi, 0) = 0 \tag{9}$$

$$\begin{aligned}
\theta'(\chi, 0) &= -1, \\
f'(\chi, \infty) &= \chi^2, \theta(\chi, \infty) = 0.
\end{aligned} \tag{10}$$

Primes in the above equations denote partial differentiation with respect to η. We note that $\chi = 0$ and $\chi = 1$ correspond to pure free and forced convection cases respectively.

For practical applications, it is usually the velocity components, friction factor and Nusselt number that are of interest. These are given by

$$u = \frac{U_\infty}{\chi^2} f'(\chi, \eta),$$

$$\nu = -\frac{\alpha Pe_x^{1/2}}{x}\left[\frac{1}{2\chi} f + \left(\frac{1-\chi}{\chi}\right)\frac{2\lambda+1}{2(2n+1)} f + (\chi - 1)\frac{2\lambda+1}{2(2n+1)}\frac{\partial f}{\partial x}\right] + \frac{\alpha Pe_x^{1/2}}{x}\frac{\eta f'}{2}\left[\frac{1}{\chi} + \left(\frac{\chi-1}{\chi}\right)\frac{2\lambda+1}{2n+1}\right]$$

$$C_{fx}\frac{2\tau_w}{\rho U_\infty^2} = \frac{2}{Ra_x}(\chi^{-3n}) Pe_x^{n/2}[f''(\chi, 0)]^n,$$

$$Nu_x = \frac{Pe_x^{1/2}\chi^{-1}}{\theta(\chi, 0)} \tag{11}$$

Numerical scheme

The numerical scheme to solve equations (8) and (9) adopted here is based on a combination of the following concepts:

(a) The boundary condition for $\eta = \infty$ is replaced by

$$f'(\chi, \eta_{\max}) = \chi^2, \theta(\chi, \eta_{\max}) = 0 \tag{12}$$

where $\eta_{\max}$ is a sufficiently large value of η at which the boundary conditions are satisfied. In the present work, a value of $\eta_{\max} = 25$ was checked to be sufficient for free stream behavior.

(b) The two-dimensional domain of interest (χ, η), is discretized with an equispaced mesh in the χ direction and another equispaced mesh in the η-direction.

(c) The partial derivatives with respect to χ and η are evaluated by the central difference approximation. The central difference approximation for the partial derivatives with respect to χ vanish when $\chi = 0$ and $\chi = 1$ which correspond to the first and the last mesh points on the χ axis, or free and forced convection respectively.

(d) Two iteration loops based on the successive substitution are used because of the nonlinearity of the equations.

(e) In each inner iteration loop, the value of χ is fixed while each of the equations (7) and (8) is solved as a linear second order boundary value problem of ODE on the η-domain. The inner iteration is continued until the nonlinear solution converges with a criterion of 10^{-6} in all cases for fixed value of χ.

(f) In the outer iteration loop, the value of χ is advanced from 0 to 1. The derivatives with respect to χ are updated after every outer iteration step. More details of the method may be obtained from Pop and Gorla (1991).

The results are affected by the number of mesh points in both directions. To obtain accurate results, a mesh sensitivity study was performed. After some trials, in the η-direction 190 mesh points were chosen whereas in the χ-direction, 51 mesh points were used. The tolerance for convergence was 10^{-6}. Increasing the mesh points to a larger value led to identical results.

Results and discussion

Numerical results for the Nusselt number are presented in Tables I-VI. To assess the accuracy of the present results, we have chosen a comparison of our results with those of Hsieh *et al.* (1993) for the case of Newtonian fluid, namely, $n = 1, H = 0$. It may be noted that the agreement between our results and the literature value is within 2-5 percent difference. We therefore conclude that our results are very accurate. The results indicate that the heat generation/absorption has considerable influence on the heat transfer rate (Nusselt number). Figures 1-4 display the variation of the local heat

	Present results		Hsieh *et al.*	
χ	$\lambda = 0.0$	$\lambda = 0.5$	$\lambda = 0.0$	$\lambda = 0.5$
1.0	0.56414	0.88601	0.5642	0.8862
0.9	0.51028	0.80144	0.5098	0.8014
0.8	0.46201	0.72637	0.4603	0.7259
0.7	0.42148	0.66417	0.4174	0.6629
0.6	0.39062	0.61883	0.3832	0.6160
0.5	0.37207	0.59411	0.3603	0.5890
0.4	0.36569	0.59202	0.3506	0.5844
0.3	0.37165	0.61195	0.3550	0.6026
0.2	0.38913	0.65136	0.3732	0.6419
0.1	0.41698	0.70701	0.4035	0.6991
0.0	0.45383	0.77584	0.4438	0.7704

Table I. Comparison of values of $-\theta'(\chi,0)$ for n=1.0 and H=0.0

H	χ	$\lambda = 0.0$	$\lambda = 0.5$	$\lambda = 1.0$	$\lambda = 2.0$
0.0	0.0	0.40692	0.85862	0.94949	1.27807
–0.4	0.5	0.49318	0.71224	0.88105	1.14799
	1.0	0.83032	1.08397	1.29155	1.63143
0.0	0.5	0.46968	0.86445	1.01902	1.36897
	1.0	0.56414	0.93070	1.12812	1.50420
0.4	0.5	0.27525	0.55150	0.75044	1.04899
	1.0	0.15871	0.63898	0.93922	1.36612

Table II. Values of $-\theta'(\chi,0)$ for $n = 0.5$

transfer rate (Nusselt number) with χ for the values of n, H and λ. As the viscosity index increases, we notice that the Nusselt number tends to decrease. This indicates that pseudoplastic ($n < 1$) fluids are associated with higher heat transfer rates when compared to dilatant ($n > 1$) fluids. As λ increases, the Nusselt number increases, thus indicating that

H	χ	$\lambda = 0.0$	$\lambda = 0.5$	$\lambda = 1.0$	$\lambda = 2.0$
0.0	0.0	0.43927	0.76178	0.98489	1.32547
–0.4	0.5	0.55656	0.85643	1.07731	1.42058
	1.0	0.83032	1.08397	1.29155	1.63143
0.0	0.5	0.39326	0.63694	0.81443	1.09045
	1.0	0.56414	0.88601	1.12796	1.50391
0.4	0.5	0.37196	0.72019	0.96653	1.33689
	1.0	0.15871	0.63898	0.93922	1.36612

Table III. Values of $-\theta'(\chi, 0)$ for $n = 0.8$

H	χ	$\lambda = 0.0$	$\lambda = 0.5$	$\lambda = 1.0$	$\lambda = 2.0$
0.0	0.0	0.45383	0.77584	1.00426	1.35033
–0.4	0.5	0.47618	0.67215	0.82569	1.07030
	1.0	0.83032	1.08397	1.29155	1.63143
0.0	0.5	0.37207	0.59411	0.75854	1.01203
	1.0	0.56414	0.88601	1.12812	1.50404
0.4	0.5	0.24693	0.50252	0.68767	0.96549
	1.0	0.15871	0.63898	0.93922	1.36612

Table IV. Values of $-\theta'(\chi, 0)$ for $n = 1.0$

H	χ	$\lambda = 0.0$	$\lambda = 0.5$	$\lambda = 1.0$	$\lambda = 2.0$
0.0	0.0	0.48134	0.80415	1.03548	1.39064
–0.4	0.5	0.45628	0.62424	0.75790	0.97593
	1.0	0.83032	1.08397	1.29155	1.63143
0.0	0.5	0.34682	0.54268	0.68737	0.91615
	1.0	0.56414	0.88601	1.12796	1.50391
0.4	0.5	0.21210	0.44229	0.61042	0.86280
	1.0	0.15871	0.63898	0.93922	1.36612

Table V. Values of $-\theta'(\chi, 0)$ for $n = 1.5$

H	χ	$\lambda = 0.0$	$\lambda = 0.5$	$\lambda = 1.0$	$\lambda = 2.0$
0.0	0.0	0.50698	0.82209	1.05505	1.41437
–0.4	0.5	0.44742	0.60252	0.72836	0.93248
	1.0	0.83032	1.08397	1.29155	1.63143
0.0	0.5	0.36003	0.52496	0.65829	0.87287
	1.0	0.56666	0.88638	1.12812	1.50404
0.4	0.5	0.19593	0.41425	0.57439	0.81494
	1.0	0.15871	0.63898	0.93922	1.36612

Table VI. Values of $-\theta'(\chi, 0)$ for $n = 2.0$

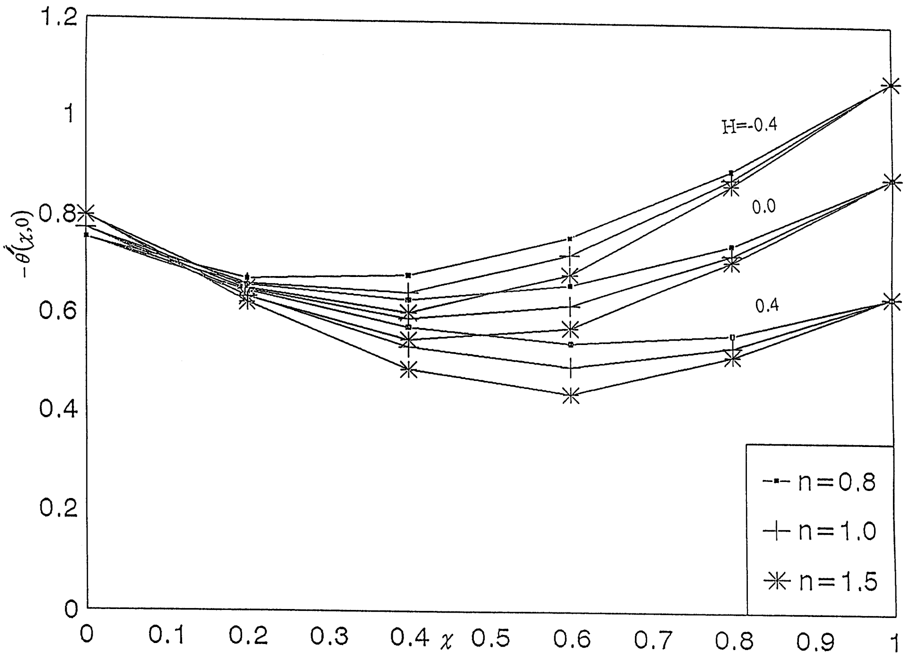

Figure 1.
Local Nusselt number versus χ with $\lambda = 0.5$

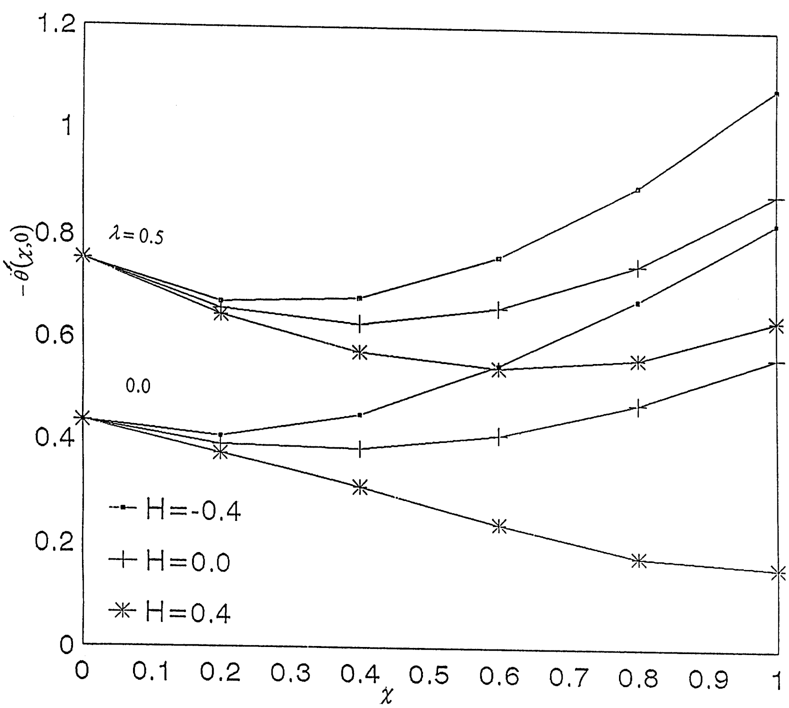

Figure 2.
Local Nusselt number versus χ with $\lambda = 0.8$

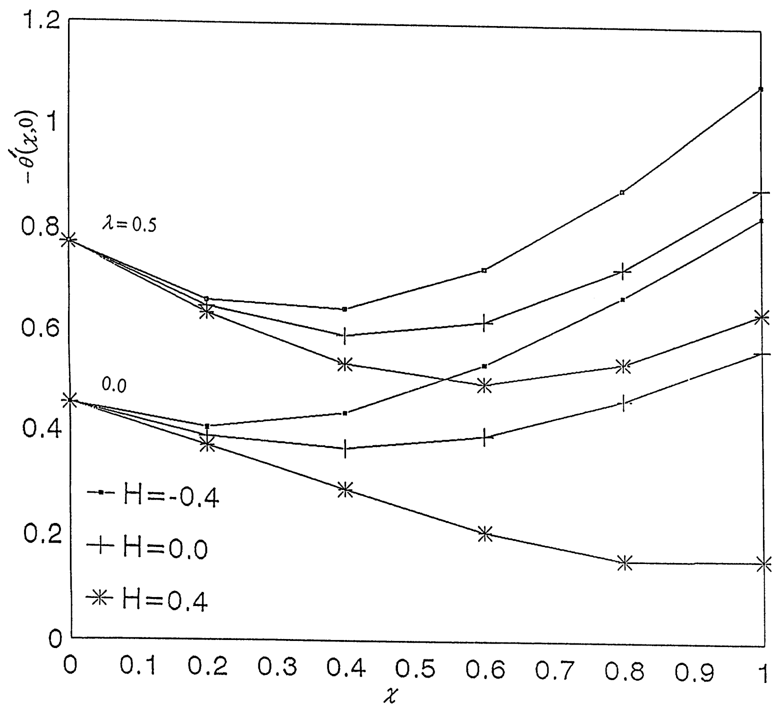

Figure 3.
Local Nusselt number versus χ with $\lambda = 1.0$

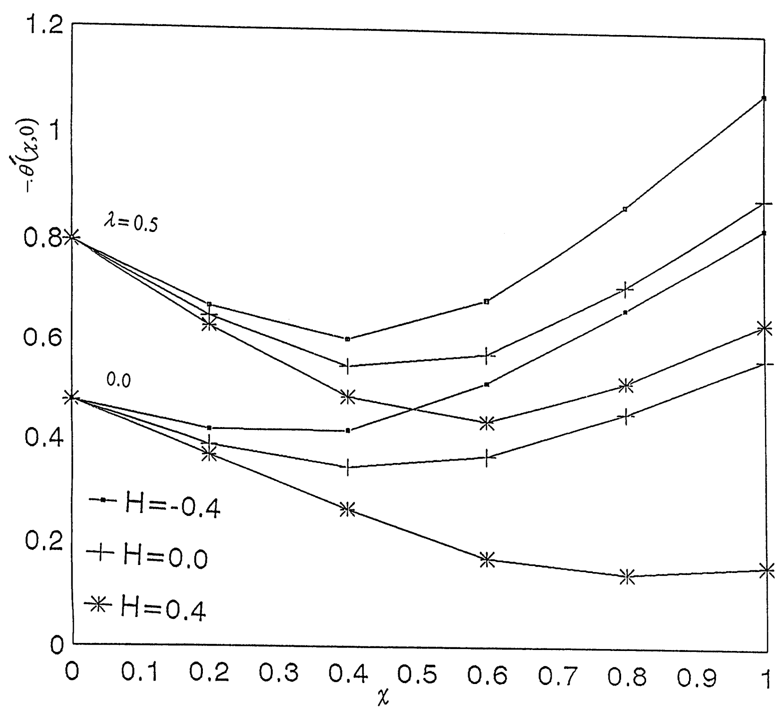

Figure 4.
Local Nusselt number versus χ with $\lambda = 1.5$

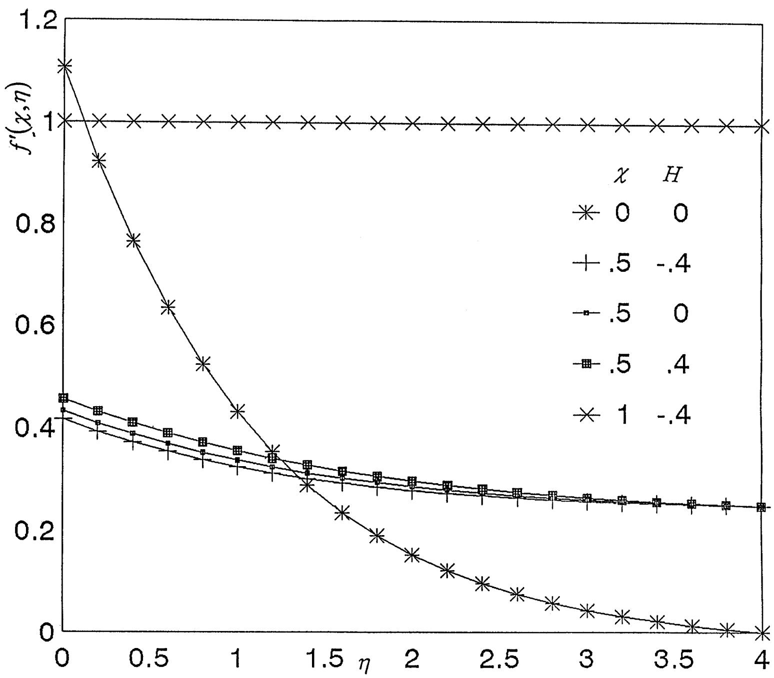

Figure 5.
Velocity profiles with $\lambda = 0.5$ and $n = 0.8$

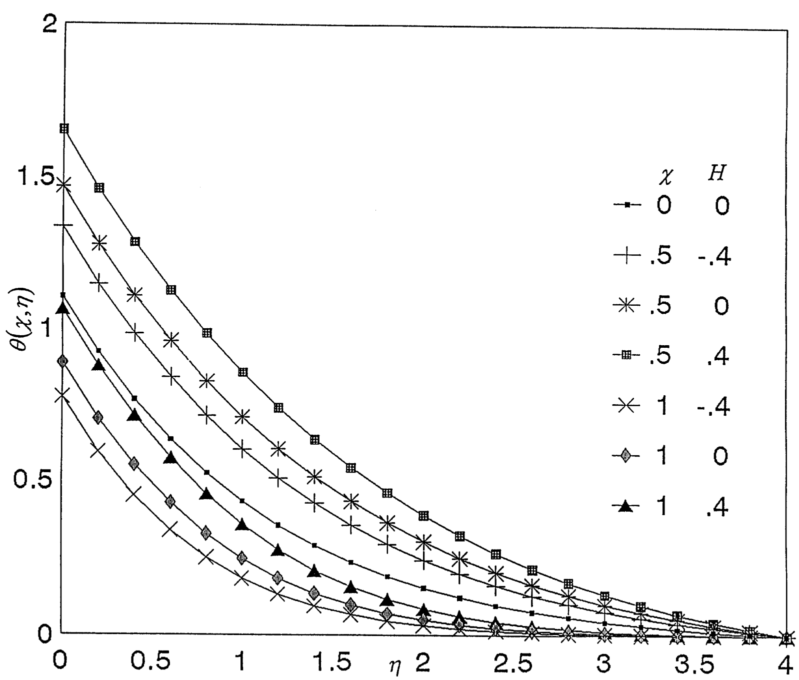

Figure 6.
Temperature profiles with $\lambda = 0.5$ and $n = 0.8$

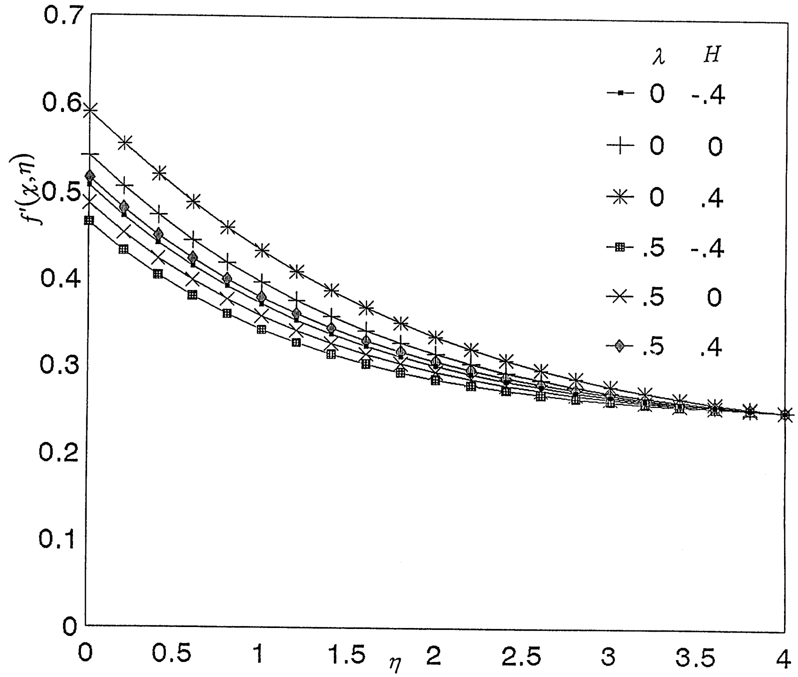

Figure 7.
Velocity profiles with $\chi = 0.5$ and $n = 0.8$

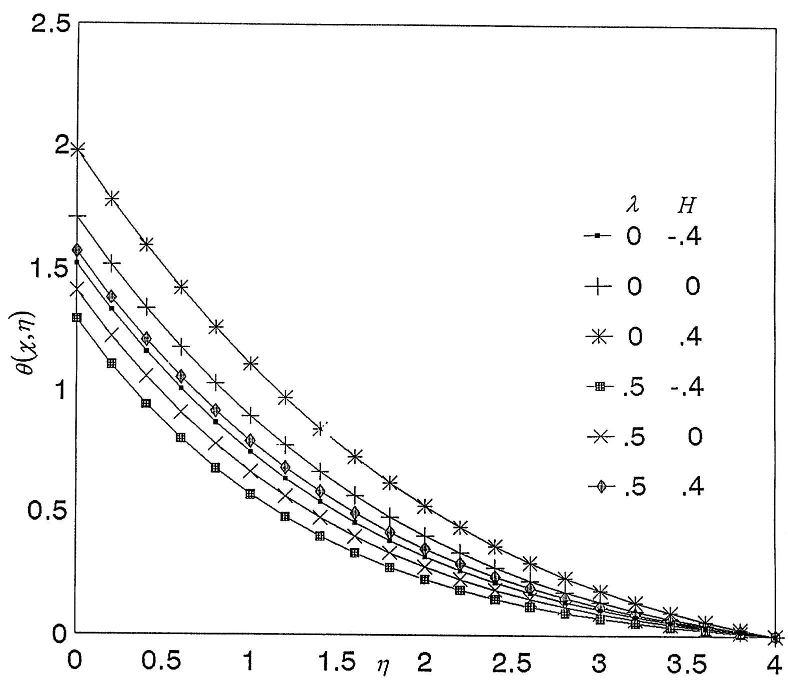

Figure 8.
Temperature profiles with $\chi = 0.5$ and $n = 0.8$

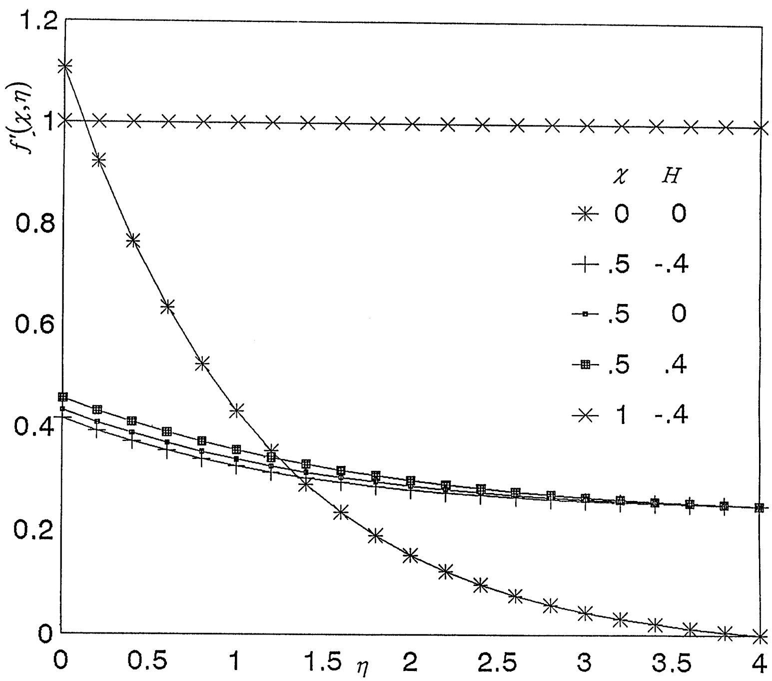

Figure 9.
Velocity profiles with $\lambda = 0.5$ and $n = 1.0$

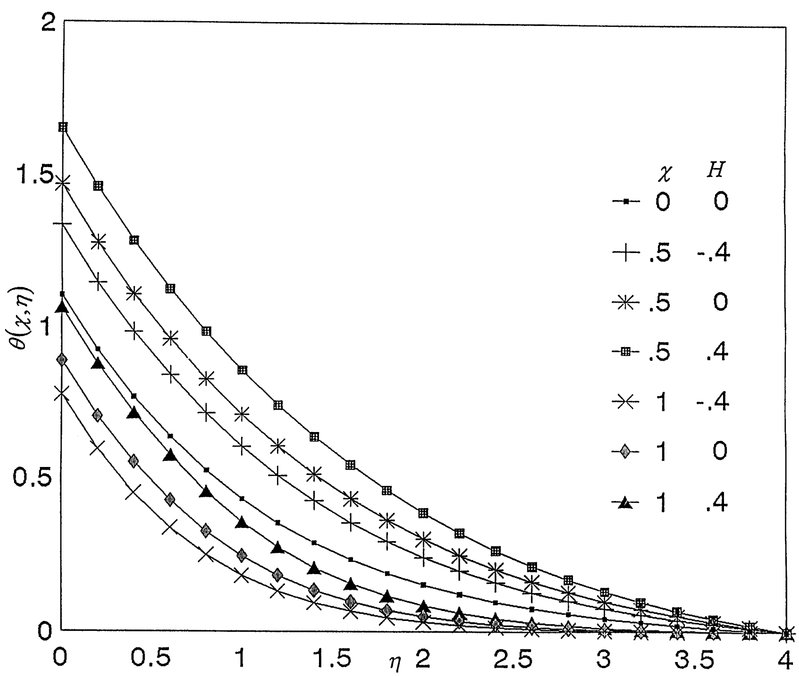

Figure 10.
Temperature profiles with $\lambda = 0.5$ and $n = 1.0$

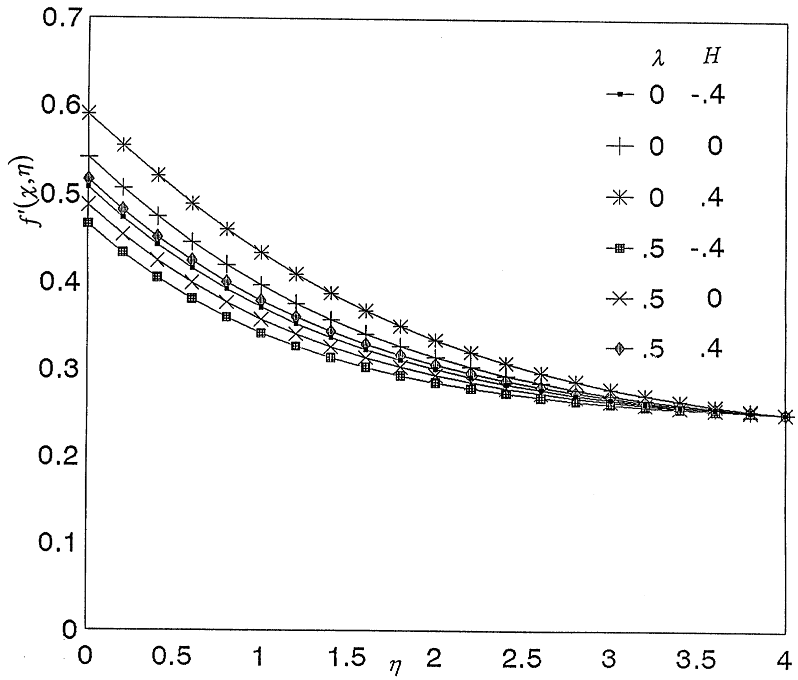

Figure 11.
Velocity profiles with $\chi = 0.5$ and $n = 1.0$

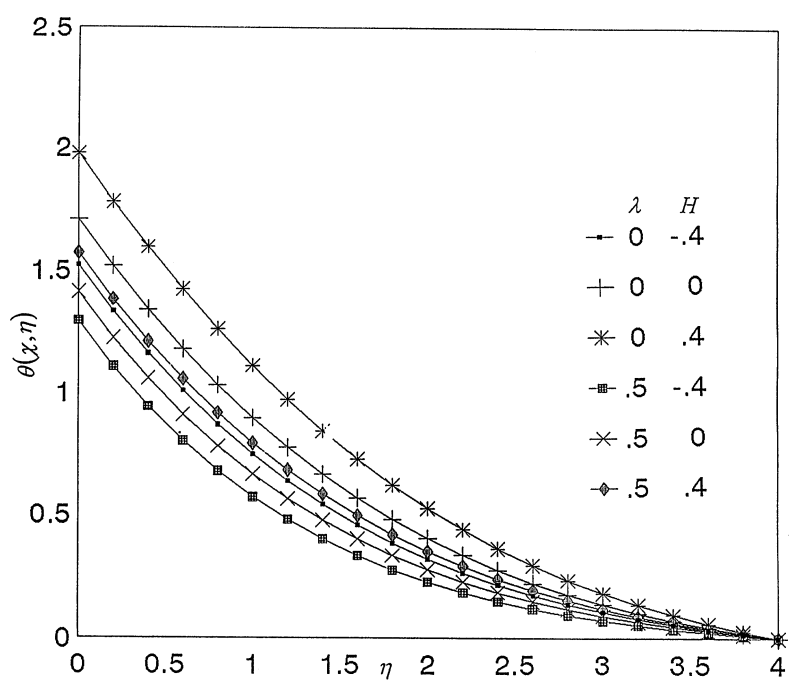

Figure 12.
Temperature profiles with $\chi = 0.5$ and $n = 1.0$

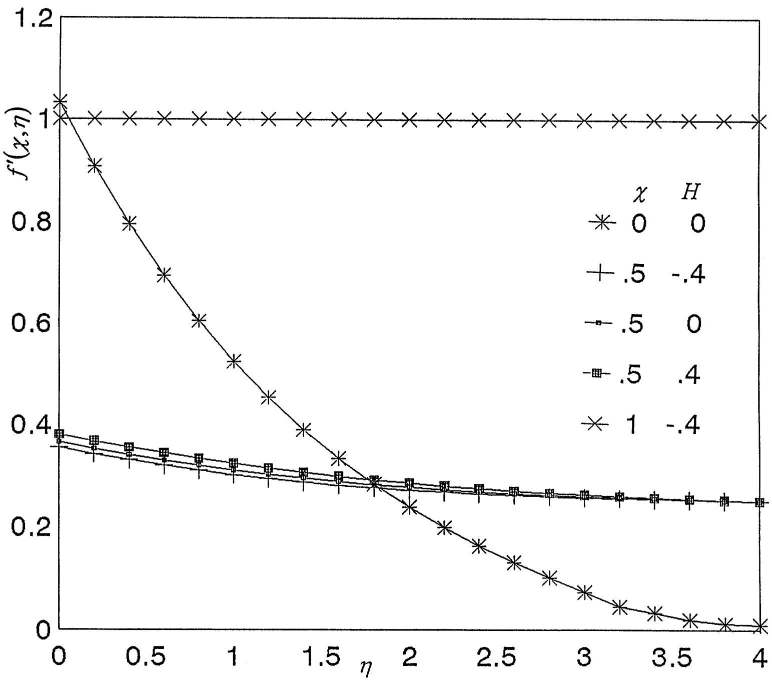

Figure 13.
Velocity profiles with $\lambda = 0.5$ and $n = 1.5$

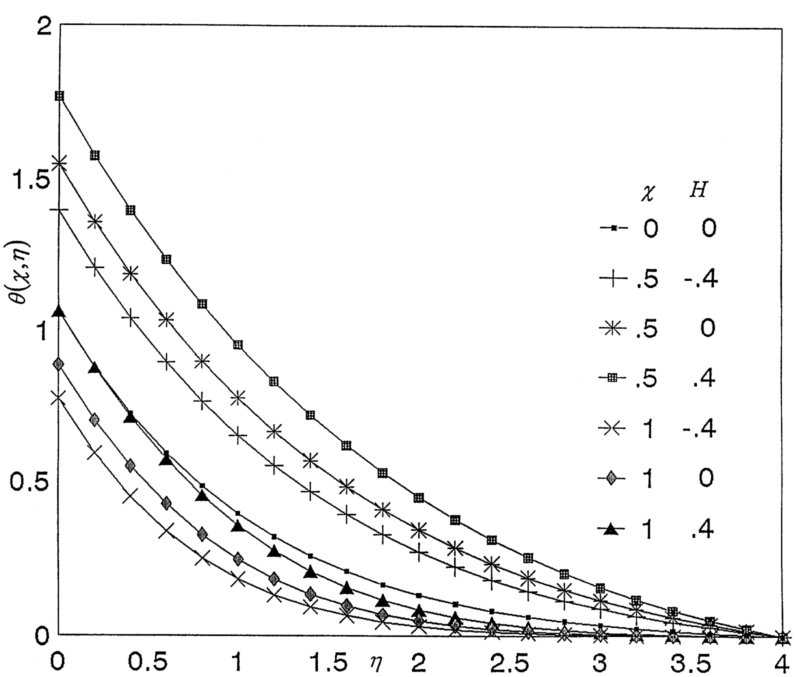

Figure 14.
Temperature profiles with $\lambda = 0.5$ and $n = 1.5$

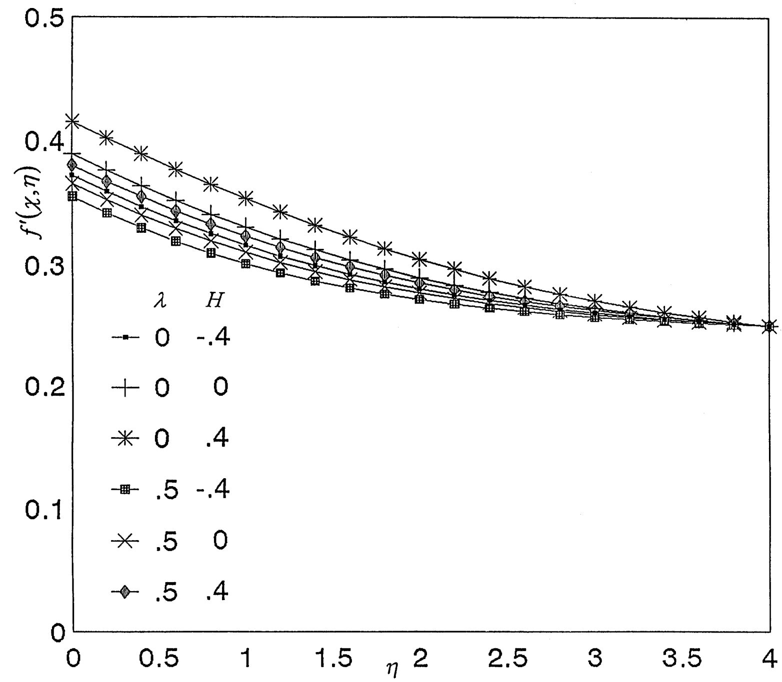

Figure 15.
Velocity profiles with $\chi = 0.5$ and $n = 1.5$

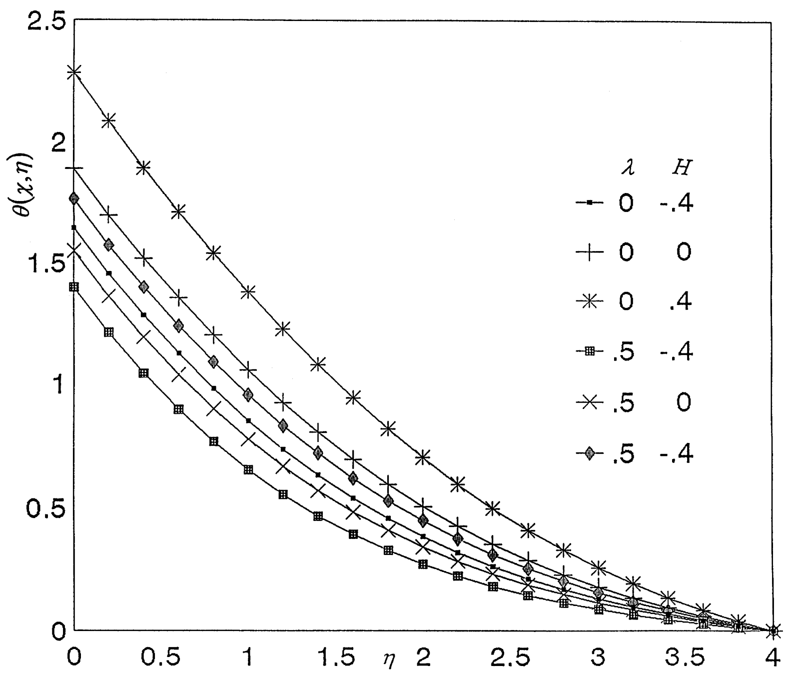

Figure 16.
Temperature profiles with $\chi = 0.5$ and $n = 1.5$

nonisothermal surfaces are associated with higher heat transfer rates than isothermal surfaces. Also, the heat generation increases, the Nusselt number decreases.

Figures 5-16 display the results for the velocity and temperature profiles. As the heat generation term increases, the velocity and temperature profiles broaden and become more uniform.

Concluding remarks

In this paper, we have presented an analysis for the problem of mixed convection from a vertical surface with variable wall heat flux and embedded in a porous medium saturated with Ostwald de-Waele type non-Newtonian fluid. The nonsimilar parameter χ is introduced. Numerical results are presented for the velocity and temperature profiles as well as Nusselt number variation with the combined convection parameter χ. The influence of internal heat generation on the surface heat transfer is examined.

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